Critical behavior in a quasifractal Ising model

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A strongly nonuniform Ising model with ferromagnetic nearest-neighbor interactions on a regular triangular lattice is considered. The interactions are assumed to be of two kinds: couplings of arbitrary large strength, distributed between points forming a Sierpiński-gasket lattice, and infinitesimally small couplings acting within all holes of the gasket. The weak interactions are, in general, allowed to vary in a hierarchical way. Using a renormalization-group method, critical properties of the system are studied. In particular, a condition for the occurrence of critical phenomena at nonzero temperatures is established. It is also shown that, in a special case, the investigated model displays nonuniversal critical properties. [S1063-651X(96)14511-6]

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I. INTRODUCTION

The effect of inhomogeneity on the critical behavior of magnetic systems has been considered in various contexts, e.g., disorder [1], coupling randomness [2], quasiperiodic structures [3], or aperiodic structures with modulated interactions on regular lattices [4]. Very special examples of nonuniform systems are models defined on fractal lattices [5]. Due to a strong nonuniformity (on all length scales), critical properties of systems in this category differ, in general, considerably from those of respective translationally invariant systems [5]. In particular, it seems to be impossible to classify fractal spin systems according to universality by a finite number of geometric parameters [6]. It has been argued that critical phenomena at nonzero temperatures cannot occur in short-interacting fractal models when their ramification order is finite [7]. One of the most known finitely-ramified self-similar (scale-invariant) systems is the nearest-neighbor (NN) Ising model on the Sierpiński-gasket (SG) lattice. In spite of the fact that this model does not display nontrivial critical phenomena, it is especially interesting because of its distinctive thermodynamic properties [8]. On the other hand, it has been shown that Ising systems defined on SG-like lattices involving higher length generators acquire gradually thermodynamic properties of the translationally invariant Ising system on the triangular lattice, as the lattice construction generators tend to infinity [9]. Clearly, the behavior of higher members of the SG family exemplifies a crossover from fractal to uniform (Euclidean) structures.

Generally, the inhomogeneity of lattice systems has two origins, i.e., the nonuniformity of lattices and variation of interactions. It is well known that both the types of inhomogeneity can affect critical properties of spin systems, compared with properties of pure (homogeneous) systems. The question of the influence of inhomogeneity on critical behavior of spin systems belongs to the most important problems of the theory of critical phenomena.

In this paper, a ferromagnetic Ising model which, in some sense, is intermediate between spin systems on fractal and translationally invariant lattices, is considered. The model involves NN interactions of two kinds, i.e., relatively large couplings acting between points which form the SG lattice, and infinitesimally small couplings distributed within all holes of the SG lattice. Additionally, the infinitesimally small interactions are parametrized by a hierarchy exponent, so as to allow them to vary in a hierarchical manner as intrahole triangles of increasing linear size are taken into account. As a consequence of extreme disproportion between strengths of interactions of the two types, the system is strongly nonuniform in all length scales. Thus, it resembles self-similar fractal systems, although it is defined on a translationally invariant lattice; so the model can be treated in some sense as intermediate between fractal and Euclidean systems. Therefore, the considered strongly nonuniform model is called here the *quasifractal* Ising model.

The presence of weak interactions inside holes of the SG lattice causes the system to be infinitely ramified. Then, at least for some values of the hierarchy exponent, the model can be expected to reveal critical phenomena at nonzero temperatures. Using here a renormalization-group (RG) method, it is shown that the system displays qualitatively different properties, according to the value of the hierarchy exponent, and a condition for the occurrence of nontrivial critical phenomena is established. For a special value of the hierarchy exponent, the system is proved to exhibit nonuniversal critical behavior. In this case, results obtained for the critical temperature and the correlation length critical exponent ν become exact as the interaction strength of weak (intrahole) interactions tends to zero. The limit passing can be regarded as a smooth Euclidean-to-fractal crossover, accomplished by enhancing the nonuniformity of the system in consequence of diminishing the interaction strength for a part of the couplings.

II. THE QUASIFRACTAL MODEL

The inhomogeneous spin Ising model considered in this paper is defined on a triangular lattice, which, as a whole, has a triangle shape (for any finite number of lattice points). The spins $\sigma_i = \pm 1$, $i = 1,2, \ldots$, are coupled by ferromagnetic NN interactions of two kinds. Interactions of the first type, K, are assumed to act only between points forming the SG lattice, as shown in Fig. 1, while interactions of the second type are distributed inside all holes of the gasket. Couplings of the latter type are taken to be infinitesimally small. In addition, these weak interactions are allowed to vary in

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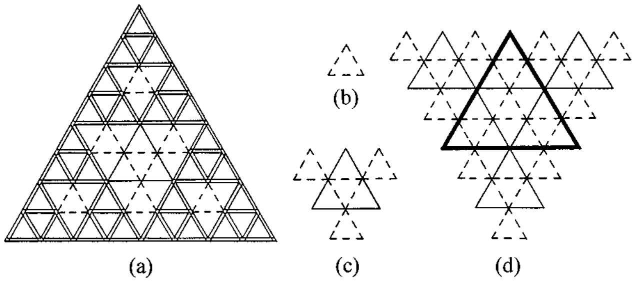

FIG. 1. Configuration of interactions in the quasifractal model. The system of linear size 2^3 (a). The intrahole sublattices of linear sizes: 2 (b), 2^2 (c), and 2^3 (d). The double lines represent interactions K, distributed on the SG sublattice. The dashed, solid, and solid bold lines represent intrahole interactions ε, ε_1, and ε_2, respectively.

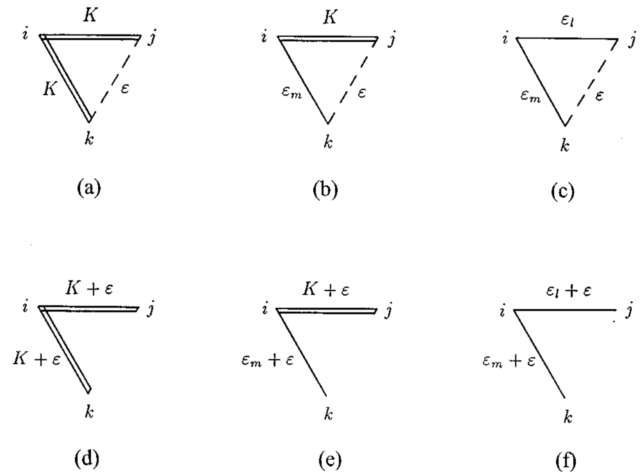

FIG. 2. (a),(b),(c) Intrahole elementary triangles with appropriate couplings [$m \geqslant 1$ in (b), and $l \geqslant 1$, $m \geqslant 1$ in (c)]. (d),(e),(f) The effect of the use of the relations (3.7) and (3.10) for local Boltzmann weights associated with respective triangles.

strength as ε^{1/μ^l}, $\varepsilon \ll 1$, $\mu \geqslant 0$, when intrahole triangles of increasing linear sizes 2^l, $l=0,1,\ldots$ (in units of the lattice constant) are taken into account (as depicted in Fig. 1). Consequently, if the parameter μ, the so-called hierarchy exponent, is nonzero, the weak couplings change in a hierarchical manner. The interactions K are assumed to be relatively large, i.e., $K \gg \varepsilon$.

Thus, the model represents strongly inhomogeneous Ising systems of nonuniform coupling structures in all length scales. The inhomogeneity originates from an extreme disproportion of strengths of the interactions K and ε_l, $l=0,1,\ldots$, as well as from their nonuniform distribution on the lattice. The strong interactions K act on the SG lattice, whereas the weak interactions ε_l, $l=0,1,\ldots$, are assumed to act inside all holes of the SG, and, for $\mu>0$, they are allowed to follow a hierarchical structure. In consequence of the special distribution of strong couplings, the model resembles a fractal system defined on the SG lattice. However, owing to the occurrence of weak couplings, the system is infinitely ramified, and can be expected to reveal critical phenomena at nonzero temperatures (at least for some values of μ) as long as $\varepsilon>0$.

Accordingly, the model presented in this paper can be used to study the question of disappearing nontrivial critical phenomena as a result of a smooth passage from infinitely to finitely ramified systems, through a smooth enhancing of the inhomogeneity of couplings. (As yet, this problem has not been investigated.) It will be shown below that the model is also useful for examining the question of how interaction inhomogeneities on all length scales influence critical properties of Ising systems, compared with pure (homogeneous) systems.

The Hamiltonian for the model is given by

$$H(\{\sigma\})/k_BT=-K\sum_{\langle i,j\rangle_{\mathrm{SG}}}\sigma_i\sigma_j-\sum_{l=0}^{\infty}\varepsilon_l\sum_{\langle i,j\rangle_H}^{(l)}\sigma_i\sigma_j, \qquad (2.1)$$

where the first summation runs over NN pairs of sites in the SG lattice, the second summation is over construction levels of intrahole triangles, and the third summation is over NN pairs of lattice points on edges of all intrahole triangles of sizes 2^l, put down in gasket holes of linear sizes 2^m with $m=l+1, m+2,\ldots$. The weak interactions ε_l are assumed to be of the form

$$\varepsilon_l=\varepsilon^{1/\mu^l}, \quad l=0,1,\ldots \qquad (2.2)$$

with $\varepsilon_0 \equiv \varepsilon$.

III. RG TREATMENT OF THE MODEL FOR $\mu \leqslant 2$

Critical properties of the quasifractal model are studied here using a RG method based on the decimation transformation. It is well known that, in cases of infinitely ramified systems, just as in the case of the system under examination here, successive applications of this transformation generate interactions of new types and, thereby, the decimation cannot be exactly performed. However, as will be argued below, the application of a renormalization transformation (RT) to the studied system with $\mu \leqslant 2$ can be simplified by modifying local Boltzmann weights [10] which involve weak interactions. The modification of these Boltzmann weights in the case of $\mu \leqslant 2$ will be shown to enable one to obtain RG equations in a closed form.

A. Boltzmann weights for $\varepsilon \ll 1$

To present the way of approximating local Boltzmann weights involving weak interactions, consider spins σ_i, σ_j, and σ_k placed in vertices of an elementary triangle in a corner of a hole of the SG [see Fig. 2(a)]. Then, the corresponding Boltzmann weight of the spin configurations $(\sigma_i,\sigma_j,\sigma_k)$ is given by

$$w_0(\sigma_i,\sigma_j,\sigma_k)=e^{K\sigma_i(\sigma_j+\sigma_k)+\varepsilon\sigma_j\sigma_k}. \qquad (3.1)$$

Since $\varepsilon \ll 1$, possible critical phenomena in the studied system can be expected at rather low temperatures, i.e., at $K \gg 1$. Hence, restricting oneself to K being very large, one gets for the spin configurations $\{+,-,-\}$ and $\{-,+,+\}$

$$w_0(\sigma_i,\sigma_j,\sigma_k)=e^{-\varepsilon}v_0(\sigma_i,\sigma_j,\sigma_k)+O(\delta\varepsilon) \quad (3.2)$$

with

$$v_0(\sigma_i,\sigma_j,\sigma_k)=e^{(K+\varepsilon)\sigma_i(\sigma_j+\sigma_k)} \quad (3.3)$$

and

$$\delta=e^{-2K}, \quad (3.4)$$

while for remaining configurations of spins at vertices of the elementary triangle

$$w_0(\sigma_i,\sigma_j,\sigma_k)=e^{-\varepsilon}v_0(\sigma_i,\sigma_j,\sigma_k). \quad (3.5)$$

Consequently, the Boltzmann weights w_0 can be approximated for $\varepsilon\ll 1$ and $K\gg 1$ by the weights v_0 (up to the factor $e^{-\varepsilon}$).

Analogously, one can express other local Boltzmann weights, associated with elementary triangles situated inside holes of the SG [see Figs. 2(b) and 2(c)]. It proves that, in the case of $\mu\leqslant 2$, all local intrahole Boltzmann weights

$$w_{l,m}(\sigma_i,\sigma_j,\sigma_k)=e^{\phi_l\sigma_i\sigma_j+\phi_m\sigma_i\sigma_k+\varepsilon\sigma_j\sigma_k}, \quad (3.6)$$

where $\phi_0=K$, $\phi_l=\varepsilon_l$, $l=1,2,\ldots$, can be written for the spin configurations $\{+,-,-\}$ and $\{-,+,+\}$ as

$$w_{m,l}(\sigma_i,\sigma_j,\sigma_k)=e^{-\varepsilon}v_{l,m}(\sigma_i,\sigma_j,\sigma_k)+O(\Delta_{l,m}\varepsilon), \quad (3.7)$$

with

$$v_{l,m}(\sigma_i,\sigma_j,\sigma_k)=e^{(\phi_l+\varepsilon)\sigma_i\sigma_j+(\phi_m+\varepsilon)\sigma_i\sigma_k}, \quad (3.8)$$

$$\Delta_{l,m}=e^{-\phi_l-\phi_m}, \quad l=0,1,\ldots, \quad m=0,1,\ldots, \quad (3.9)$$

where $\Delta_{0,0}\equiv\delta$, and, for remaining configurations of spins connected with appropriate intrahole triangles, can be expressed as

$$w_{l,m}(\sigma_i,\sigma_j,\sigma_k)=e^{-\varepsilon}v_{l,m}(\sigma_i,\sigma_j,\sigma_k). \quad (3.10)$$

Accordingly, for $\varepsilon\ll 1$, the local Boltzmann weights $w_{l,m}$ can be represented in the case of $\mu\leqslant 2$ by the weights $v_{l,m}$ (up to the multiplicative constant $e^{-\varepsilon}$). Such a representation is approximate for two configurations of spins assigned to a given elementary triangle, and is exact for the remaining six spin configurations. The resulting approximations are highly accurate for those weights $w_{l,m}$ which involve at least one interaction K (note that $\Delta_{l,0}\to 0$, $l=0,1,\ldots$, and $\Delta_{0,m}\to 0$, $m=0,1,\ldots$, as $K\to\infty$), whereas, for weights $w_{l,m}$ with $l\neq 0$ and $m\neq 0$, associated with elementary triangles that are not adjacent to hole borders, they appear to be much less precise. However, as will be shown below, when the RT is performed, the approximation for $l\neq 0$ and $m\neq 0$ turns out to be also reasonable (provided that $\mu\leqslant 2$).

It should be pointed out that the representation of the local Boltzmann weights $w_{l,m}$ by the modified weights $v_{l,m}$ [according to Eqs. (3.6)–(3.10)] corresponds to a removal of the weak interaction ε, attributed to a given elementary intrahole triangle, and to respective changes of the remaining two interactions ascribed to the triangle [as shown in Figs. 2(d), 2(e), and 2(f)]. Clearly, such a modification of local Boltzmann weights resembles the Migdal-Kadanoff moving-bond RG procedure [11].

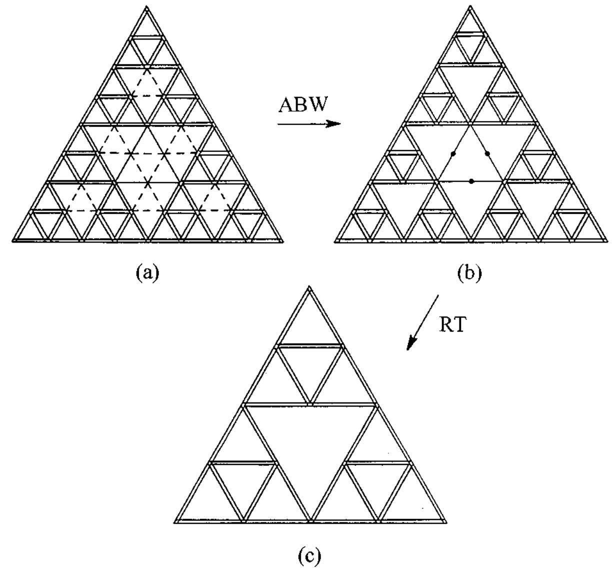

FIG. 3. The effect of using the RT combined with the relations (3.7) and (3.10). (a) The initial quasifractal system of a finite linear size (the couplings are indicated in the same way as in Fig. 1). (b) The system after an application of the approximation for local Boltzmann weights, according to relations (3.7) and (3.10). The black circles indicate spins over which the decoration-iteration transformation is to be carried out. (c) The system after applying the RT.

B. RG equations for $\mu\leqslant 2$

In cases of $\mu\leqslant 2$, possible critical properties of the system can be studied by applying the relations (3.7) and (3.10). Then, the RT can indepedently be performed for the subsystem on the SG lattice and for intrahole subsystems (associated with holes of the SG lattice), by combining the decimation transformation and the decoration-iteration transformation [12]. It should be pointed out that the use of Eqs. (3.7) and (3.10) for intrahole subsystems leads to a reduction of the decimation transformation to the decoration-iteration transformation, as shown in Fig. 3. Consequently, employing Eqs. (3.7), (3.10), and applying the decimation transformation for the subsystem on the SG lattice and the decoration-iteration transformation for intrahole subsystems (see Fig. 3), yields the relations (to orders δ^2 and ε)

$$K_1^{(1)}=K-\delta^2, \quad (3.11)$$

$$K_2^{(1)}=K-\delta^2+\varepsilon, \quad (3.12)$$

$$\varepsilon_l^{(1)}=\varepsilon_{l+1}^2, \quad l=0,1,\ldots, \quad (3.13)$$

where the superscript specifies the particular iteration of the RT and subscripts in Eqs. (3.11) and (3.12) label strong interactions of different kinds. Interactions of both kinds are between NN pairs of spins, but the couplings $K_1^{(1)}$ act along edges of the largest triangle of the SG lattice, while the cou-

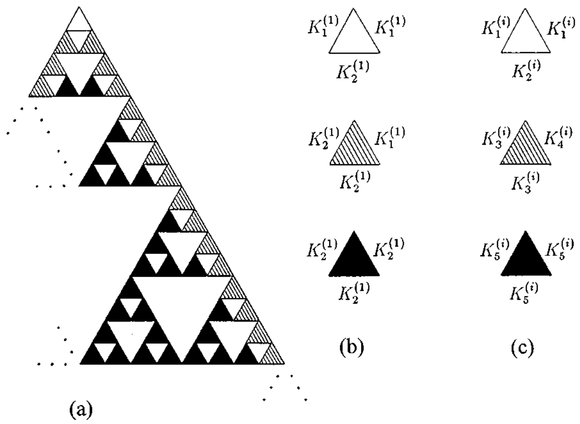

FIG. 4. The SG subsystem of a renormalized quasifractal system. (a) The distribution of renormalized strong couplings. The white, hatched, and black upward-pointed elementary triangles are associated with different sets of NN interactions. (b) The case of $i=1$ (the system after the first application of the RT). All interactions $K_1^{(1)}$ (connected with white and hatched triangles) act along borders of the SG (i.e., along edges of the largest triangles of the SG). (c) The case of successive iterations of the RT ($i\geqslant 2$). The interactions $K_1^{(i)}$ (associated only with the white triangle) and the interactions $K_4^{(i)}$ (ascribed to hatched triangles) are distributed along the borders of the SG.

plings $K_2^{(1)}$ act along the remaining edges of the SG lattice, as shown in Figs. 4(a) and 4(b). Thus, in the case of $\varepsilon\ll 1$, the first use of the RT generates NN interactions of two types, but it does not generate any two-point long-range couplings nor multipoint interactions.

As a result of the second iteration of the RT [with the use of relations (3.7) and (3.10) for the renormalized interactions (3.11), (3.12), and (3.13)], there appear NN couplings of five types [see Figs. 4(b) and 4(c)]. For $K\gg 1$ and $\varepsilon\ll 1$, the respective RG relations have the form

$$K_1^{(2)}=K_1^{(1)}-(\delta_2^{(1)})^2, \tag{3.14}$$

$$K_2^{(2)}=K_2^{(1)}-\tfrac{1}{2}[(\delta_1^{(1)})^2+(\delta_2^{(1)})^2]+\varepsilon^{(1)}, \tag{3.15}$$

$$K_3^{(2)}=K_2^{(1)}-\tfrac{1}{4}[3(\delta_1^{(1)})^2+(\delta_2^{(1)})^2]+\varepsilon^{(1)}, \tag{3.16}$$

$$K_4^{(2)}=K_1^{(1)}-\tfrac{1}{4}[5(\delta_1^{(1)})^2-(\delta_2^{(1)})^2], \tag{3.17}$$

$$K_5^{(2)}=K_2^{(1)}-(\delta_2^{(1)})^2+\varepsilon^{(1)}, \tag{3.18}$$

$$\varepsilon_l^{(2)}=(\varepsilon_{l+1}^{(1)})^2, \quad l=0,1,\ldots, \tag{3.19}$$

where $\delta_n^{(1)}=\exp(-2K_n^{(1)})$, $n=1,2,\ldots,5$, and $\varepsilon^{(1)}\equiv\varepsilon_0^{(1)}$.

Subsequent iterations of the RT [combined with relations (3.7) and (3.10) for renormalized couplings] do not lead to the appearance of couplings of new types; so the resulting RG equations take a closed form. Generally, the RG equations at the $(i+1)$th iteration of the RT ($i>2$) can be written as (for $K\gg 1$ and $\varepsilon\ll 1$)

$$K_1^{(i+1)}=K_1^{(i)}-\tfrac{1}{2}[(\delta_3^{(i)}\delta_4^{(i)}/\delta_1^{(i)})^2+\delta_2^{(i)}\delta_3^{(i)}\delta_4^{(i)}/\delta_1^{(i)}], \tag{3.20}$$

$$\begin{aligned}K_2^{(i+1)}=K_3^{(i)}+K_4^{(i)}-K_1^{(i)}-\tfrac{1}{2}[&(\delta_1^{(i)})^2+(\delta_3^{(i)})^2\\ &-(\delta_3^{(i)}\delta_4^{(i)}/\delta_1^{(i)})^2-\delta_2^{(i)}\delta_3^{(i)}\delta_4^{(i)}/\delta_1^{(i)}\\ &+2\delta_1^{(i)}\delta_2^{(i)}\delta_3^{(i)}/\delta_4^{(i)}]+\varepsilon^{(i)},\end{aligned} \tag{3.21}$$

$$\begin{aligned}K_3^{(i+1)}=K_5^{(i)}-\tfrac{1}{4}[&2(\delta_3^{(i)})^2+(\delta_3^{(i)})^4/(\delta_5^{(i)})^2+(\delta_3^{(i)}\delta_4^{(i)}/\delta_5^{(i)})^2]\\ &+\varepsilon^{(i)},\end{aligned} \tag{3.22}$$

$$\begin{aligned}K_4^{(i+1)}=K_3^{(i)}+K_4^{(i)}-K_5^{(i)}-\tfrac{1}{4}[&6(\delta_5^{(i)})^2-(\delta_3^{(i)})^4/(\delta_5^{(i)})^2\\ &-(\delta_3^{(i)}\delta_4^{(i)}/\delta_5^{(i)})^2],\end{aligned} \tag{3.23}$$

$$K_5^{(i+1)}=K_5^{(i)}-(\delta_5^{(i)})^2+\varepsilon^{(i)}, \tag{3.24}$$

$$\varepsilon_l^{(i+1)}=(\varepsilon_{l+1}^{(i)})^2, \quad l=0,1,\ldots, \tag{3.25}$$

where $\delta_n^{(i)}=\exp(-2K_n^{(i)})$, $n=1,2,\ldots,5$, and $\varepsilon^{(i)}\equiv\varepsilon_0^{(i)}$. The division of the renormalized NN couplings into five classes is a consequence of the specific nonuniform structure of the system. Obviously, the numbers of couplings included in particular classes are different. The numbers of the interactions $K_1^{(i)}$ and $K_2^{(i)}$, $i>2$, do not depend on the size of the system and amount to $N_1=6$ and $N_2=3$, respectively. In a finite renormalized system, which contains 3^k, $k=2,3,\ldots$, renormalized strong interactions (acting on the SG lattice), there are $N_3=3(2^k-4)$ couplings of the type $K_3^{(i)}$, $N_4=3(2^{k-1}-2)$ couplings $K_4^{(i)}$, and $N_5=3^k[1-3(2/3)^{k-1}]-9$ couplings $K_5^{(i)}$. Accordingly, the relative number of interactions from a given class (the number of interactions belonging to a given class per the total number of strong interactions) is nonzero in the thermodynamic limit only for the interactions of the type $K_5^{(i)}$.

It is seen from Eqs. (2.2), (3.13), (3.19), and (3.25) that, for $\mu\leqslant 2$, the weak interactions $\varepsilon_l^{(i)}$, $i=1,2,\ldots$, do not increase under the RT, i.e., $\varepsilon_l^{(i)}\geqslant\varepsilon_l^{(i+1)}$ $l=0,1,\ldots$, $i=1,2,\ldots$. However, the relations (3.13), (3.19), and (3.25) have been derived by taking advantage of the approximation (3.7), which, for $l\neq 0$ and $m\neq 0$, is not so well justifiable as in the case of $l=0$ and/or $m=0$. Nevertheless, leading contributions to any renormalized weak interactions (two-point or multipoint) obtained in cases of $\mu\leqslant 2$ at the ith iteration of a RT without the use of the approximation (3.7) can be at most of order $(\varepsilon_l^{(i)})^2$ (for appropriate l). Accordingly, in cases of $\mu\leqslant 2$, the application of the approximate relation (3.7) does not affect the leading contributions to renormalized interactions as ε is infinitesimally small. Thus, in these cases, the use of the approximation (3.7) at each iteration step of the RT proves to be plausible, provided that $\varepsilon\ll 1$. Futhermore, the RG equations (3.20)–(3.25) become exact as $\varepsilon\to 0$ and $K\to\infty$.

IV. CRITICAL PROPERTIES OF THE MODEL

A. The case of $\mu<2$

It follows from Eqs. (3.13), (3.19), and (3.25) that, for $\mu<2$, the renormalized weak couplings tend to zero under the successive application of the RT. Thus, the fixed point solution to Eq. (3.25) is given by $\varepsilon_l^*\equiv\varepsilon_l^{(\infty)}=0$,

$l=0,1,\ldots$. Hence, in the case of $\mu<2$, Eqs. (3.20)–(3.25) reduce for sufficiently large i (and for $K\gg1$) to RG equations for the Ising model on the pure SG lattice [7]. Consequently, the model with $\mu<2$ does not reveal critical properties at nonzero temperatures, despite the existence in the system intrahole interactions ε_l, $l=0,1,\ldots$.

It should be noted that Eqs. (3.20)–(3.25) have been derived for $K\gg1$. Thus, these relations do not hold for rather small values of K, and cannot be used for studying properties of the system at high temperatures. Clearly, the quasifractal model with $\mu<2$ does not reveal critical behavior also at high temperatures, as it does not display long-range ordering at low nonzero temperatures.

B. The case of $\mu=2$

In this special case of $\mu=2$, one has $\varepsilon_l^{(i)}=(\varepsilon_{l+1}^{(i)})^2$, $l=0,1,\ldots$, $i=1,2,\ldots$, so Eq. (3.25) implies that

$$\varepsilon_l^{(i+1)}=\varepsilon_l^{(i)},\quad l=0,1,\ldots\ . \tag{4.1}$$

This means that, for $\mu=2$, the RT leaves the weak interactions unchanged, i.e., $\varepsilon_l^{(i)}=\varepsilon_l$, $l=0,1,\ldots$, $i=1,2,\ldots$. It can be easily seen that, in this case, Eqs. (3.20)–(3.25) have a nontrivial fixed point solution given by

$$K_1^*=\infty, \tag{4.2}$$

$$K_2^*=-\tfrac{1}{4}\ln\varepsilon+\tfrac{3}{4}\varepsilon, \tag{4.3}$$

$$K_3^*=-\tfrac{1}{4}\ln\varepsilon+\tfrac{1}{4}\varepsilon, \tag{4.4}$$

$$K_4^*=\infty, \tag{4.5}$$

$$K_5^*=-\tfrac{1}{4}\ln\varepsilon, \tag{4.6}$$

$$\varepsilon_l^*=\varepsilon_l,\quad l=0,1,\ldots, \tag{4.7}$$

with $\delta_1^*/\delta_4^*=1$, where $\delta_n^*=\exp(-2K_n^*)$. (Note that K_2^*, K_3^*, and K_5^* have been determined to order ε.) It should be pointed out that, although $K_1^*=\infty$, $K_4^*=\infty$, the fixed point given by Eqs. (4.2)–(4.7) can indeed be considered a nontrivial one. This follows from the fact that the relative numbers of the interactions $K_1^{(i)}$ and $K_4^{(i)}$ (i.e., the ratios of the numbers of $K_1^{(i)}$ and $K_4^{(i)}$ to the total number of renormalized strong interactions) tend to zero as the thermodynamic limit is approached. Then, in this limit, the couplings $K_1^{(i)}$ and $K_4^{(i)}$ do not affect thermodynamic properties of the system, or more precisely, do not contribute to the density of the free energy of the system. Obviously, since the interactions $K_1^{(i)}$ and $K_4^{(i)}$ act along edges of the largest triangle of the SG lattice, they are much less influenced by the nonuniformity of the system, compared to the remaining couplings. This is why the interactions $K_1^{(i)}$ and $K_4^{(i)}$ behave in a quite different manner than do the couplings $K_2^{(i)}$, $K_3^{(i)}$, and $K_5^{(i)}$, as $i\to\infty$.

Taking into account that Eq. (3.24) does not involve interactions of other types than $K_5^{(i)}$, one immediately finds for $\varepsilon\ll1$ the critical temperature parameter K_c:

$$K_c=-\tfrac{1}{4}\ln\varepsilon. \tag{4.8}$$

Thus, the critical temperature for the model with $\mu=2$ is greater than zero, as long as $\varepsilon>0$. It can readily be proved that the derivative matrix, obtained by linearizing the RG equations (3.20)–(3.25), has at the fixed point (4.2)–(4.7) one eigenvalue greater than 1. This eigenvalue, determined simply by $\lambda=(\partial K_5^{(i+1)}/\partial K_5^{(i)})_{K_5^*}$, has to order ε the form

$$\lambda=1+\tfrac{1}{4}\varepsilon. \tag{4.9}$$

Hence, the critical exponent ν characterizing the behavior of the correlation length at criticality ($\nu=\ln b/\ln\lambda$ with $b=2$ being the length rescaling factor) is given, for $\varepsilon\ll1$, by

$$\nu=\frac{\ln2}{4\varepsilon}. \tag{4.10}$$

As a result of the dependence of this index on the strength of the weak interactions, critical properties of the system with $\mu=2$ are nonuniversal. Note that, in the limit $\varepsilon\to0$, one recovers values of K_c and ν for the Ising model on the pure SG lattice, i.e., one obtains $K_c=\infty$ and $\nu=\infty$ [7].

C. The case of $\mu>2$

It follows from Eqs. (2.2), (3.13), (3.19), and (3.25) that, in the case of $\mu=2$, the weak interactions $\varepsilon_l^{(i)}$, $l=0,1,\ldots$, increase under the RT (even for arbitrary small but nonzero ε). Therefore, from a certain iteration step i of the RT, the renormalized coupling $\varepsilon_l^{(i)}$, $l=0,1,\ldots$, cannot be regarded as weak. Then, the approximate relation (3.7) for the Boltzmann weights, associated with intrahole renormalized interactions, and the RG equations (3.20)–(3.25) are no longer valid. Thus, the problem of the existence of critical points reflecting strong inhomogeneity of the system with $\mu>2$ cannot be studied by employing a simple RG method that separately treats strong and weak interactions, as the RG approach used above. Consequently, the analysis of critical properties of the model with $\mu>2$ appears to be substantially more complicated than the one presented for the case of $\mu\leqslant2$, and it will not be carried out here.

V. DISCUSSION

The quasifractal Ising model introduced in this paper is defined on a triangular lattice which, for a finite number of lattice points, has the shape of a triangle. The model involves NN ferromagnetic interactions of two types: strong and infinitesimally small. The strong couplings act between spins located at points which form a SG lattice of external borders coinciding with borders of the triangular lattice. The weak interactions are assumed to act inside all holes of the gasket. These couplings are additionally parametrized by the exponent $\mu\geqslant0$, in such a manner that they are allowed to vary in a hierarchical manner as larger and larger triangles within holes of the gasket are taken into account. The case of $\mu=0$ concerns the system in which all weak interactions are of an equal strength. It should be noted that the strong inhomogeneity of the model is a consequence of extremal variation of interaction strengths in all length scales rather than a result of dilution of couplings and/or lattice sites. Thus, as long as $\varepsilon>0$, there always exist linkage paths across the whole system (for each value of μ), in contrast to dilute systems in which infinite clusters do not occur when the concentration of couplings is smaller than a threshold value, or when the concentration of site defects exceeds a critical value [1].

According to the argumentation of the preceding section, if $\mu<2$, the system does not undergo critical phenomena at finite temperatures, although it is infinitely ramified. Thus, the weak interactions are unable to establish long-range order at nonzero temperatures when $\mu<2$. Then, the behavior of the system with $\varepsilon \ll 1$ and $\mu<2$ exhibits some similarities to the behavior of the Ising system on the pure SG lattice. Hence, there must exist, for a given $\mu<2$, a threshold value of ε, above which the system displays critical phenomena at nonzero temperature (note that, for $\varepsilon=K$ and $\mu=0$, the model reduces to the homogeneous Ising model on the triangular lattice). Obviously, the threshold value of ε is expected to depend on μ.

In the case of $\mu=2$, the quasifractal model exhibits critical behavior at nonzero temperatures (provided that the strength ε of weak interactions is greater than zero). The critical exponent ν found for the model with $\mu=2$ turns out to depend on ε, and thereby the critical properties of the system are in this case nonuniversal. As $\varepsilon \to 0$, the behavior of the system shows a smooth crossover to the behavior of the Ising model on the pure SG lattice. As distinct from the previously studied fractal-to-Euclidean crossover [9], which arises by changing the construction of the underlying SG-like lattices towards the triangular lattice, the crossover demonstrated in the quasifractal model is caused by the disappearance of weak interactions, i.e., by intensifying the nonuniformity of the system, defined on the uniform triangular lattice.

Finally, consider the case of $\mu>2$. In this case, the successive use of the RT leads (at least for some initial iterations) to increasing the renormalized weak interactions, so they can no longer be treated as being small. Such an effect of homogenization of the studied system is similar to the effect of restoration of macroscopic isotropy, observed in various fractal models with microscopic anisotropy [13]. As a consequence of the homogenization, the problem of investigating critical phenomena in the quasifractal model with $\mu>2$ appears to be much more involved than in the case of $\mu \leqslant 2$. Obviously, the model with $\mu>2$ must reveal nontrivial critical properties, since such properties are displayed by the model in the case of $\mu=2$ (note that each interaction ε_l, $l=0,1,\ldots$, is stronger for $\mu>2$ than for $\mu=2$).

In conclusion, there is a simple criterion of the existence of nontrivial critical phenomena in the quasifractal model: the model with infinitesimally small ε exhibits critical properties at nonzero temperatures if the hierarchy exponent $\mu \geqslant 2$. An open question is the determination of conditions of the occurrence of nontrivial critical behavior in the nonuniform model when ε is not infinitesimally small.

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